15.3.2) Polar Geometry and Double Integrals

A circle with radius r has area πr^2. A *sector* of the circle with central angle θ (measured in radians) has area $\frac{1}{2}\theta r^2$. (Notice that the full circle is obtained when $\theta = 2\pi$, in which case the formula $\frac{1}{2}\theta r^2$ gives us $\frac{1}{2}(2\pi)r^2 = \pi r^2$.)

An *annulus* is a region bounded by two concentric circles. (A more casual term for this is a *ring*.) The area of the annulus is the difference of the areas of the two circles: If the inner circle has radius r_1 and the outer circle has radius r_2, then the area of the annulus is $\pi(r_2)^2 - \pi(r_1)^2 = \pi[(r_2)^2 - (r_1)^2]$. A *sector* of the annulus with central angle θ (measured in radians) has area $\frac{1}{2}\theta(r_2)^2 - \frac{1}{2}\theta(r_1)^2 = \frac{1}{2}\theta[(r_2)^2 - (r_1)^2]$. (Notice that the full annulus is obtained when $\theta = 2\pi$, in which case the formula $\frac{1}{2}\theta[(r_2)^2 - (r_1)^2]$ gives us $\frac{1}{2}(2\pi)[(r_2)^2 - (r_1)^2] = \pi[(r_2)^2 - (r_1)^2]$.)

The formula for the area of a sector of an annulus can be rewritten as follows:

$$\frac{1}{2}\theta[(r_2)^2 - (r_1)^2] = \frac{1}{2}\theta(r_2 + r_1)(r_2 - r_1) = \frac{r_1 + r_2}{2}(r_2 - r_1)\theta.$$

$\frac{r_1 + r_2}{2}$ is the average of the inner radius and outer radius of the annulus; we may refer to this as the *average radius* of the annulus, and we may denote it as r_{av}.

$r_2 - r_1$ is the difference of the outer and inner radii; we may refer to this as the *width* of the annulus, and we may denote it as w.

Thus, the area of a sector of an annulus is $r_{av} w\theta$ (in other words, average radius times width times central angle measure).

A **polar rectangle** is either a sector of a circle or a sector of an annulus. In polar coordinates, it is $\{(r,\theta) \mid r \in [a,b],\ \theta \in [\alpha,\beta]\}$, where $0 \le a < b$ and $\alpha < \beta \le \alpha + 2\pi$. This set may also be written as $[a,b] \times [\alpha,\beta]$.

- If $a = 0$, it is a sector of a circle.
- If $a > 0$, it is a sector of an annulus.

The central angle of the polar rectangle $[a,b] \times [\alpha,\beta]$ is $\beta - \alpha$.

If our polar rectangle is a circle sector (i.e., if $a = 0$), then its radius is b, so its area is $\frac{1}{2}(\beta - \alpha)b^2$.

If our polar rectangle is an annulus sector (i.e., if $a > 0$), then its average radius is $\frac{a+b}{2}$ and its width is $b - a$, so its area is $\frac{a+b}{2}(b - a)(\beta - \alpha)$.

To compute the area of the polar rectangle $[a,b] \times [\alpha,\beta]$, we may *always* use the formula $\frac{a+b}{2}(b-a)(\beta-\alpha)$, since if $a = 0$, this reduces to the other formula, $\frac{1}{2}(\beta-\alpha)b^2$.

In constructing a Riemann Sum, we partition the polar rectangle $[a,b] \times [\alpha,\beta]$ into polar subrectangles, by partitioning the interval $[a,b]$ into n subintervals, each of length $\Delta r = \frac{b-a}{n}$, and by partitioning the interval $[\alpha,\beta]$ into n subintervals, each of length $\Delta\theta = \frac{\beta-\alpha}{n}$. We denote the subintervals of $[a,b]$ as $I_1, I_2, \ldots, I_n$, and we denote the subintervals of $[\alpha,\beta]$ as $J_1, J_2, \ldots, J_n$. Pairing each of the former with each of the latter gives us n^2 subrectangles, which we denote as $[R_{i,j}]_{i=1,\ldots,n}^{j=1,\ldots,n}$

For each subrectangle $R_{i,j}$, let $r_{i,j}$ denote its average radius. Then the area of the subrectangle is $r_{i,j}\Delta r\Delta\theta$. When we apply the limit as $n \to \infty$, this becomes $r\,dr\,d\theta$, also referred to as dA.

CAUTION: Students often forget to include the factor r in dA. In Cartesian coordinates, $dA = dx\,dy$, but in polar coordinates, $dA = r\,dr\,d\theta$, **not** just $dr\,d\theta$.

Thus, if D is the polar rectangle $[a,b] \times [\alpha,\beta]$, then $\iint_D f(x,y)\,dA =$

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

If D is the Type II polar region $\{(r,\theta) \mid \theta \in [\alpha,\beta],\ h_1(\theta) \le r \le h_2(\theta)\}$, then $\iint_D f(x,y)\,dA =$

$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$